Sharp Integrability for Brownian Motion in Parabola-shaped Regions

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Abstract

We study the sharp order of integrability of the exit position of Brownian motion from the planar domains $\mathcal{P}_\alpha = \{(x, y) \in \mathbb{R} \times \mathbb{R} \colon x > 0,\ |y| < Ax^\alpha\}$, $0 < \alpha < 1$. Together with some simple good-λ type arguments, this implies the order of integrability for the exit time of these domains; a result first proved for $\alpha = 1/2$ by Bañuelos, DeBlassie and Smits [1] and for general α by Li [9]. A sharp version of this result is also proved in higher dimensions.

Contents

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1 Introduction

For γ with $0 < \gamma \leq \pi$, we denote by Γ_γ the right circular cone in $\mathbb{R}^n$ of angle γ. We let $\{B_t : t \geq 0\}$ be n-dimensional Brownian motion and denote by E_x and P_x the expectation and the probability associated with this motion starting at x. We write $T_\gamma = \inf\{t > 0 : B_t \notin \Gamma_\gamma\}$, so that T_γ is the first exit time of the Brownian motion from Γ_γ. In 1977, Burkholder [3] found the sharp order of integrability of T_γ. More precisely, he showed the existence of a critical constant $p(\gamma, n)$ (given explicitly in terms of zeros of certain hypergeometric functions) such that, for $z \in \Gamma_\gamma$,

$$E_z T_\gamma^{p/2} < \infty, \tag{1.1}$$

if and only if $p < p(\gamma, n)$. In particular, for a cone in two dimensions (a case that had already been solved in [14]) $E_z T_\gamma^{p/2} < \infty$ if and only if $p < \pi/(2\gamma)$. Notice that by making the angle of the cone arbitrarily small we can make p arbitrarily large.

The cone in two dimensions can be thought of as the domain above the graph of the function $y = a|x|$. It is then natural to study the order of integrability of exit times from other unbounded regions, in particular parabolas. Since any parabola is contained in a cone of arbitrarily small angle, the exit time for a parabola has finite moments of all orders. On the other hand, by comparing with rectangles, it is also easy to show that the exit time is not exponentially integrable. Bañuelos, DeBlassie and Smits showed in [1] that if $\tau_{\mathcal{P}}$ is the exit time of the Brownian motion from the parabola $\mathcal{P} = \{(x, y) \colon x > 0,\ |y| < A\sqrt{x}\}$ and $z \in \mathcal{P}$, then there exist positive constants A_1 and A_2 such that

$$\begin{aligned} -A_1 \ &\leq\ \liminf_{t\to\infty} t^{-\frac{1}{3}} \log\left[P_z\{\tau_{\mathcal{P}} > t\}\right] \\ &\leq\ \limsup_{t\to\infty} t^{-\frac{1}{3}} \log\left[P_z\{\tau_{\mathcal{P}} > t\}\right]\ \leq\ -A_2. \end{aligned} \tag{1.2}$$

Thus if $a < 1/3$, then $E_z[\exp(b\tau_{\mathcal{P}}^a)] < \infty$ for each $b > 0$ and, if $a > 1/3$, then $E_z[\exp(b\tau_{\mathcal{P}}^a)] = \infty$ for each $b > 0$. This result was extended to all dimensions and to other unbounded regions by W. Li [9]. More recently, M. Lifshits and Z. Shi [10] found the limit $-l$ of $t^{-\frac{1}{3}} \log\left[P_z\{\tau_{\mathcal{P}} > t\}\right]$, as $t \to \infty$. Then, if $b < l$ then $E_z[\exp(b\tau_{\mathcal{P}}^{1/3})] < \infty$ and if $b > l$ then $E_z[\exp(b\tau_{\mathcal{P}}^{1/3})] = \infty$, (it is not known whether or not $E_z[\exp(l\tau_{\mathcal{P}}^{1/3})]$ is finite or infinite), so that Lifshits and Shi determine the sharp order of integrability of the exit time $\tau_{\mathcal{P}}$ (see the proof of Theorem 2). Their result holds for

the more general parabolic regions studied in [9] in any dimension. In [2], M. van den Berg used the sharp results of Lifshits and Shi to obtain analogues of these results for the Dirichlet heat kernel. Finally, similar results have been obtained recently by D. DeBlassie and R. Smits [5] for "twisted parabolas" in two dimensions.

In the case of the cone Γ_γ in $\mathbb{R}^n$, the sharp order of integrability of the exit position B_{T_γ} is known. For any domain D in $\mathbb{R}^n$ we write $B^*_{\tau_D}$ for the maximum distance of Brownian motion from the origin up to the exit time τ_D for D. More precisely, we set

$$B^*_{\tau_D} = \sup\{|B_t|\colon 0 \le t < \tau_D\}.$$

Then, by Burkholder's inequality (Theorem 2.1 in [3]), for any finite, positive p there are constants $C^1_{p,n}$ and $C^2_{p,n}$ such that

$$C^1_{p,n} E_z \left[n\tau_D + |z|^2\right]^{p/2} \le E_z[B^*_{\tau_D}]^p \le C^2_{p,n} E_z \left[n\tau_D + |z|^2\right]^{p/2}. \tag{1.3}$$

Thus it follows from (1.1) that

$$E_z\,[B^*_{T_\gamma}]^p < \infty \tag{1.4}$$

if and only if $p < p(\gamma, n)$. We always have $|B_{T_\gamma}| \le B^*_{T_\gamma}$. On the other hand, if $p > 1$ and

$$E_z|B_{T_\gamma}|^p < \infty \tag{1.5}$$

then by Doob's maximal inequality (1.4) holds, and hence so does (1.1). This gives the sharp order of integrability $\pi/2\gamma$ of B_{T_γ} in two dimensions. For $\mathbb{R}^n$, $n \ge 3$, since T_γ is finite a.s., it is a consequence of Burkholder's Theorem 2.2 in [3] that $E_z[B^*_{T_\gamma}]^p$ is finite if $E_z|B_{T_\gamma}|^p$ is finite and this gives the sharp order of integrability of B_{T_γ} in all dimensions. The results of Burkholder generated considerable interest amongst probabilists and analysts. In particular, M. Essén and K. Haliste [6] used harmonic measure techniques to obtain some generalizations of Burkholder's results.

The problem of obtaining the sharp order of integrability of the exit position $B_{\tau_\mathcal{P}}$ for the parabola $\mathcal{P}$, and of the related random variable $B^*_{\tau_\mathcal{P}}$, suggest themselves – a problem that we address in this paper. As we shall see below, the order of integrability of $B^*_{\tau_\mathcal{P}}$ and $\tau_\mathcal{P}$ are, as in the case of cones, also closely related. We consider, as in [9] and [10], more general regions in $\mathbb{R}^n$ of the form

$$\mathcal{P}_\alpha = \{(x, Y) \in \mathbb{R} \times \mathbb{R}^{n-1}\colon x > 0,\, |Y| < Ax^\alpha\}, \tag{1.6}$$

with $0 < \alpha < 1$ and $A > 0$. (The case $\alpha = 1$ is the cone for which, as described above, we know everything.) We write τ_α for the exit time from $\mathcal{P}_\alpha$.

We begin with a very simple result which shows that the random variables $B_{\tau_\mathcal{P}}$ and $B^*_{\tau_\mathcal{P}}$ share the same integrability properties. More precisely,

Theorem 1. *Suppose that a and b are positive constants and that $z \in \mathcal{P}_\alpha$. Then $E_z\left[\exp[b\,|B_{\tau_\alpha}|^a]\right] < \infty$ if and only if $E_z\left[\exp[b\,(B^*_{\tau_\alpha})^a]\right] < \infty$.*

Naturally, we will need to estimate the distribution functions of $B_{\tau_\mathcal{P}}$ and $B^*_{\tau_\mathcal{P}}$ for large t in the manner of (1.2). The probability that the Brownian motion exits a parabola-shaped domain $\mathcal{P}_\alpha$ outside the ball of radius t, that is $P_z\{|B_{\tau_\alpha}| > t\}$, is the harmonic measure of that part of the boundary of $\mathcal{P}_\alpha$ lying outside the ball $B(0,t)$ of radius t, taken w.r.t. the domain $\mathcal{P}_\alpha$. The larger quantity $P_z\{B^*_{\tau_\alpha} > t\}$ is the harmonic measure of the intersection of $\mathcal{P}_\alpha$ with the sphere of radius t taken w.r.t. the intersection of $\mathcal{P}_\alpha$ with the ball $B(0,t)$. In fact, $1 - P_z\{B^*_{\tau_\alpha} > t\}$ is the probability of exiting $\mathcal{P}_\alpha$ without ever exiting the ball $B(0,t)$. These interpretations of the distribution functions of B_{τ_α} and $B^*_{\tau_\alpha}$ facilitate the use of some well-known and quite precise estimates of harmonic measure.

Our result for parabola-shaped regions in the plane provides a complete solution to the problem and we state it separately.

Theorem 2. *For the exit position B_{τ_α} from the parabola-shaped domain $\mathcal{P}_\alpha$ in the plane,*

$$\lim_{t\to\infty} t^{\alpha-1}\log\left[P_z\{|B_{\tau_\alpha}| > t\}\right] = -\frac{\pi}{2A(1-\alpha)}. \tag{1.7}$$

Furthermore,

$$E_z\left[\exp\left[b\,|B_{\tau_\alpha}|^{1-\alpha}\right]\right] < \infty \tag{1.8}$$

if and only if

$$b < \frac{\pi}{2A(1-\alpha)}.$$

Then, $E_z\left[\exp[b\,|B_{\tau_\alpha}|^a]\right]$ is integrable for each positive b if $a < 1-\alpha$, and is not integrable for any positive b if $a > 1-\alpha$.

We note that we can determine whether or not $\exp[b\,|B_{\tau_\alpha}|^a]$ is integrable in all cases, including the critical case $a = 1-\alpha$, $b = \pi/\left[2A(1-\alpha)\right]$.

For parabola-shaped regions in $\mathbb{R}^n$, it is proved in [9] that

$$
\begin{aligned}
(1.9) \qquad -B_1 \;\le\; & \liminf_{t\to\infty} t^{\frac{\alpha-1}{\alpha+1}} \log\left[P_z\{\tau_\alpha > t\}\right] \\
\le\; & \limsup_{t\to\infty} t^{\frac{\alpha-1}{\alpha+1}} \log\left[P_z\{\tau_\alpha > t\}\right] \;\le\; -B_2
\end{aligned}
$$

for two positive constants B_1 and B_2 depending on A, on α and on the dimension. In [10], the limit of $t^{\frac{\alpha-1}{\alpha+1}} \log\left[P_z\{\tau_\alpha > t\}\right]$ is shown to exist and its value is determined explicitly. For comparison purposes we observe that if $n = 2$, $\alpha = 1/2$ and $A = 1$, the case of the parabola in the plane, the results in [10] give

$$
(1.10) \qquad \lim_{t\to\infty} t^{-\frac{1}{3}} \log\left[P_z\{\tau_{1/2} > t\}\right] = -\frac{3\pi^2}{8},
$$

while it follows from Theorem 2 that

$$
(1.11) \qquad \lim_{t\to\infty} t^{-\frac{1}{2}} \log\left[P_z\{|B_{\tau_{1/2}}| > t\}\right] = -\pi.
$$

We stated Theorem 2 in terms of limits of logs of distributions to draw a parallel with the previously cited work on exit times. However, our results are sharper than that, as we obtain sharp estimates for the distribution itself, (see Proposition 1 below).

We prove an extension of Theorem 2 to parabola-shaped regions in higher dimensions.

Theorem 3. *We let λ_1 be the smallest eigenvalue for the Dirichlet Laplacian in the unit ball of $\mathbb{R}^{n-1}$. For the exit position B_{τ_α} from the parabola-shaped region $\mathcal{P}_\alpha$ of (1.6), and for $z \in \mathcal{P}_\alpha$,*

$$
(1.12) \qquad \lim_{t\to\infty} t^{\alpha-1} \log\left[P_z\{|B_{\tau_\alpha}| > t\}\right] = -\frac{\sqrt{\lambda_1}}{A(1-\alpha)}.
$$

Furthermore,

$$
E_z\left[\exp\left[b\,|B_{\tau_\alpha}|^{1-\alpha}\right]\right]
$$

is finite if $b < \sqrt{\lambda_1}/\left[A(1-\alpha)\right]$ and is infinite if $b > \sqrt{\lambda_1}/\left[A(1-\alpha)\right]$.

Our estimates on the distribution function of B_{τ_α} in higher dimensions are not sufficiently precise to determine whether or not $\exp\left[b\,|B_{\tau_\alpha}|^a\right]$ is integrable in the critical case $a = 1-\alpha$, $b = \sqrt{\lambda_1}/\left[A(1-\alpha)\right]$. This is one reason why we state the two dimensional result separately as Theorem 2. Moreover, though the method we use to obtain the distribution estimates

for B_{τ_α} in the planar case are relatively standard in complex analysis, it forms the general outline of the method used to obtain lower bounds for the distribution in higher dimensions. For this reason also, it seems helpful to present the two dimensional case separately.

A large part of this paper is devoted to adapting the conformal mapping techniques that have led to such precise harmonic measure estimates in planar domains to the parabola-shaped regions $\mathcal{P}_\alpha$ in $\mathbb{R}^n$. The so-called 'Carleman method' can be adapted to estimate harmonic measure in these domains from above, as we do in Section 4.1. All else being equal, harmonic measure is generally largest in the most symmetric case, and so one would expect the upper bounds given by the Carleman method to be reasonably precise. However, we were not able to use the Carleman method to obtain the lower bounds for harmonic measure that we need to determine the exact order of exponential integrability of the exit position B_{τ_α}. For this, we adapt the conformal mapping techniques used to prove Theorem 2. Our parabola-shaped domains being symmetric, we can rewrite the distribution estimates for B_{τ_α} as a distribution estimate in the corresponding planar parabola-shaped domain, but at the cost of having to deal with a Bessel-type operator rather than the Laplacian. Conformal invariance is lost, but the conformal mapping techniques can still be made to work with considerably more effort. An outline of our method for obtaining these relatively precise estimates from below for the distribution of the exit position in parabola-shaped regions can be found at the beginning of Section 4.2.

Burkholder's inequality (1.3), as outlined earlier, allows one to deduce the integrability properties of the random variable $B^*_{T_\alpha}$ for the cone Γ_α from those for the exit time T_α for the cone, and vice versa. In the case of parabola-shaped domains, we can partially pass from integrability results for the exit position to integrability results for the exit time and conversely. The proof provides a partial explanation of the connection between the two critical exponents, $1-\alpha$ in the case of the exit position and $(1-\alpha)/(1+\alpha)$ in the case of the exit time.

Theorem 4. *We may deduce from*

$$E_z\left[\exp[b_1\,(B^*_{\tau_\alpha})^{1-\alpha}]\right] < \infty, \tag{1.13}$$

with b_1 positive, that there is some positive b_2 for which

$$E_z\left[\exp\left[b_2\,{\tau_\alpha}^{\frac{1-\alpha}{1+\alpha}}\right]\right] < \infty. \tag{1.14}$$

Conversely, we may deduce from (1.14), with b_2 positive, that (1.13) holds for some positive b_1.

2 Proofs of Theorems 1 and 4

Proof of Theorem 1. Clearly $|B_{\tau_\alpha}| \le B^*_{\tau_\alpha}$ and hence if $E_z\left[\exp[b\,(B^*_{\tau_\alpha})^a]\right] < \infty$, then $E_z\left[\exp[b\,|B_{\tau_\alpha}|^a]\right] < \infty$, with the same a and b.

We turn to the proof of the converse. As observed in the Introduction, $E_z\tau_\alpha^p < \infty$, for each finite p. By (1.3) we also have $E_z(B^*_{\tau_\alpha})^p < \infty$ for p finite. By Doob's maximal inequality,

$$E_z(B^*_{\tau_\alpha})^p \le \left(\frac{p}{p-1}\right)^p E_z|B_{\tau_\alpha}|^p, \quad \text{for } 1 < p < \infty.$$

Thus there exists a p_0 such that for all $p > p_0$,

$$E_z(B^*_{\tau_\alpha})^p \le 4E_z|B_{\tau_\alpha}|^p. \tag{2.1}$$

We choose an integer k_0, depending on p_0 and a, such that $p_0 < ka$ for all $k > k_0$. Then

$$\begin{aligned}
E_z\left[\exp[b\,(B^*_{\tau_\alpha})^a]\right] &= 1 + \sum_{k=1}^{k_0}\frac{b^k}{k!}E_z(B^*_{\tau_\alpha})^{ak} + \sum_{k=k_0+1}^{\infty}\frac{b^k}{k!}E_z(B^*_{\tau_\alpha})^{ak} \\
&\le 1 + \sum_{k=1}^{k_0}\frac{b^k}{k!}E_z(B^*_{\tau_\alpha})^{ak} + 4\sum_{k=k_0+1}^{\infty}\frac{b^k}{k!}E_z|B_{\tau_\alpha}|^{ak} \\
&\le 1 + \sum_{k=1}^{k_0}\frac{b^k}{k!}E_z(B^*_{\tau_\alpha})^{ak} + 4\,E_z\left[\exp[b\,|B_{\tau_\alpha}|^a]\right] \\
&< \infty,
\end{aligned}$$

which proves the theorem. □

Proof of Theorem 4. We argue as in the proof of the classical good-λ inequalities, see [3] for example. Suppose (1.14) holds for some $b_2 > 0$. We set $\beta = 1+\alpha$ and note that $0 < \beta/2 < 1$. Then

$$P_z\{B^*_{\tau_\alpha} > t\} \;\le\; P_z\{B^*_{\tau_\alpha} > t,\ \tau_\alpha \le t^\beta\} + P_z\{\tau_\alpha > t^\beta\}.$$

For the second term (1.14), together with a Chebyshev style argument, gives

$$P_z\{\tau_\alpha > t^\beta\} \le C\exp\left[-b_2 t^{\beta(\frac{1-\alpha}{1+\alpha})}\right] = C\exp\left[-b_2 t^{1-\alpha}\right]. \tag{2.2}$$

Take $t > 2|z|$, so that $t - |z| > t/2$. By our assumption on t and by scaling,

$$\begin{aligned}
(2.3)\ P_z\{B^*_{\tau_\alpha} > t,\ \tau_\alpha \le t^\beta\} \quad &\le \quad P_z\left\{\sup_{0\le s<\tau_\alpha} |B_s - z| > \frac{t}{2}, \tau_\alpha \le t^\beta\right\} \\
&\le \quad P_z\left\{\sup_{0\le s<t^\beta} |B_s - z| > \frac{t}{2}\right\} \\
&= \quad P_0\left\{\sup_{0\le s<t^\beta} |B_s| > \frac{t}{2}\right\} \\
&= \quad P_0\left\{\sup_{0\le s<1} |B_s| > \frac{1}{2}\, t^{1-\beta/2}\right\} \\
&\le \quad C\exp\left[-\frac{1}{8}\, t^{2-\beta}\right] = C\exp\left[-\frac{1}{8}\, t^{1-\alpha}\right].
\end{aligned}$$

In the last inequality we used the well-known fact (see [3], inequality (2.15)) that

$$P_0\left\{\sup_{0\le s<1} |B_s| > \lambda\right\} \le C\exp\left[-\lambda^2/2\right].$$

It now follows from (2.2) and (2.3) that

$$P_z\{B^*_{\tau_\alpha} > t\} \le C\exp\left[-2b_1 t^{1-\alpha}\right],$$

for all large t, where $2b_1 = \min\{1/8, b_2\}$. Since

$$E_z\left[\exp[b\,(B^*_{\tau_\alpha})^{1-\alpha}]\right] = 1 + b\int_0^\infty e^{bt} P_z\{B^*_{\tau_\alpha} > t^{1/(1-\alpha)}\}\,dt,$$

we obtain (1.13).

Conversely, suppose that (1.13) holds, so that

$$P_z\{B^*_{\tau_\alpha} > t\} \le C\exp\left[-b_1 t^{1-\alpha}\right]. \tag{2.4}$$

Let $\beta = 1/(1+\alpha)$. We assume, as we may, that t is very large and set

$$\tilde{\mathcal{P}}(t) = \mathcal{P}_\alpha \cap B(0, t^\beta)$$

and write $\tilde\tau$ for its exit time. Clearly $\tilde\tau \le \tau_\alpha$ and

$$P_z\{\tau_\alpha > t\} \ \le\ P_z\{\tau_\alpha > t, \tau_\alpha = \tilde\tau\} + P_z\{\tau_\alpha > t, \tau_\alpha > \tilde\tau\}. \tag{2.5}$$

Let us denote the ball centered at 0 and of radius r in $\mathbb{R}^{n-1}$ by $B_{n-1}(0,r)$ and the exit time of Brownian motion from it by $\tau_{B_{n-1}(0,r)}$. We recall that for $t > 1$,

$$P_0\{\tau_{B_{n-1}(0,1)} > t\} \le C\exp\left[-\lambda_1 t\right],$$

where C is a constant independent of t and λ_1 is the first Dirichlet eigenvalue of $B_{n-1}(0,1)$. By our definition of the region $\tilde{\mathcal{P}}(t)$,

$$
\begin{aligned}
P_z\{\tau_\alpha > t, \tau_\alpha = \tilde\tau\} &\leq P_z\{\tilde\tau > t\} \\
&\leq P_0\{\tau_{B_{n-1}(0,t^{\alpha\beta})} > t\} \\
&= P_0\{\tau_{B_{n-1}(0,1)} > t^{1-2\alpha\beta}\} \\
&\leq C \exp\left[-\lambda_1\, t^{1-2\alpha\beta}\right], \\
&= C \exp\left[-\lambda_1\, t^{\frac{1-\alpha}{1+\alpha}}\right],
\end{aligned} \tag{2.6}
$$

where we used scaling for the first equality above. On the other hand, using (2.4),

$$
\begin{aligned}
P_z\{\tau_\alpha > t, \tau_\alpha > \tilde\tau\} &\leq P_z\{B^*_{\tau_\alpha} > t^\beta\} \\
&\leq C \exp\left[-b_1 t^{\beta(1-\alpha)}\right]. \\
&= C \exp\left[-b_1 t^{\frac{1-\alpha}{1+\alpha}}\right].
\end{aligned} \tag{2.7}
$$

The estimates (2.5)–(2.7) prove that (1.14) may be deduced from (1.13). This completes the proof of Theorem 4. $\square$

We note that our argument above proves that if $E_z\left[\exp[b_2\, {\tau_\alpha}^a]\right] < \infty$ for some $0 < a < 1$ and $b_2 > 0$, then $E_z\left[\exp[b_1\,(B^*_{\tau_\alpha})^{2a/(1+a)}]\right] < \infty$ for some b_1 depending on b_2. Conversely, if $E_z\left[\exp[b_1\,(B^*_{\tau_\alpha})^a]\right] < \infty$ for some a and b_1, then

$$
E_z\left[\exp[b_2\, {\tau_\alpha}^{a/(a+2\alpha)}]\right] < \infty
$$

for some b_2 depending on b_1. Thus by Theorems 2 and 3 we see that

$$
E_z\left[\exp[b\, {\tau_\alpha}^p]\right] < \infty \tag{2.8}
$$

for some $b > 0$ if and only

$$
p \leq \frac{1-\alpha}{1+\alpha},
$$

as was already proved in [1] for $\alpha = 1/2$ and in [9] for general α in (0, 1).

3 Parabola-shaped domains in the plane

3.1 Harmonic measure estimates

For the moment we restrict ourselves to two dimensions. The key estimate for the distribution function for B_{τ_α}, of which the limit (1.7) is a direct con-

sequence, is Proposition 1. As we shall see below, we may suppose without loss of generality that $z = z_0$, where z_0 is the point $(1, 0)$.

Proposition 1. *There are constants C_1 and C_2 depending only on α and A such that, as $t \to \infty$,*

$$\begin{aligned} C_1 \exp\left[-\frac{\pi}{2A(1-\alpha)} t^{1-\alpha}\right] &\leq P_{z_0}\{|B_{\tau_\alpha}| > t\} \\ &\leq C_2 \exp\left[-\frac{\pi}{2A(1-\alpha)} t^{1-\alpha} + \mathrm{o}(t^{1-\alpha})\right]. \end{aligned}$$

The first step in the proof of Proposition 1 is to show that there is a negligible difference between $P_{z_0}\{|B_{\tau_\alpha}| > t\}$ and $P_{z_0}\{B^1_{\tau_\alpha} > t\}$, where for $z = (x, Y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ we write z^1 for x, the projection onto the first coordinate. This will follow from a simple estimate that holds in all dimensions.

Lemma 1. *For the exit position B_{τ_α} of Brownian motion from $\mathcal{P}_\alpha$ in $\mathbb{R}^n$, and for sufficiently large t,*

$$P_{z_0}\{B^1_{\tau_\alpha} > t\} \;\leq\; P_{z_0}\{|B_{\tau_\alpha}| > t\} \;\leq\; P_{z_0}\{B^1_{\tau_\alpha} > t - A^2 t^{2\alpha-1}\}.$$

Proof. For fixed t positive, we write $S(0, t)$ for the sphere center 0 and radius t and we write $x(t)$ for the common first coordinate of the points of intersection of the boundary of $\mathcal{P}_\alpha$ with $S(0, t)$. Then

$$P_{z_0}\{|B_{\tau_\alpha}| > t\} = P_{z_0}\{B^1_{\tau_\alpha} > x(t)\}. \tag{3.1}$$

We also have, for all sufficiently large t,

$$t - A^2 t^{2\alpha-1} < x(t) < t. \tag{3.2}$$

The upper bound is clear. For the lower bound, we observe that any point (x, Y) with $x = t - A^2 t^{2\alpha-1}$ and $|Y| = A(t - A^2 t^{2\alpha-1})^\alpha$ lies inside the sphere $S(0, t)$. Indeed,

$$\begin{aligned} (t - A^2 t^{2\alpha-1})^2 &+ A^2(t - A^2 t^{2\alpha-1})^{2\alpha} - t^2 \\ &= A^4 t^{4\alpha-2} - 2A^2 t^{2\alpha} + A^2(t - A^2 t^{2\alpha-1})^{2\alpha} \\ &\leq A^4 t^{4\alpha-2} - 2A^2 t^{2\alpha} + A^2 t^{2\alpha} \\ &= A^2 t^{2\alpha}(A^2 t^{2(\alpha-1)} - 1) \\ &< 0, \end{aligned}$$

where we use the assumption that $0 < \alpha < 1$ and note that $0 < t - A^2 t^{2\alpha-1} < t$. ∎

Since $\left(t - A^2 t^{2\alpha-1}\right)^{1-\alpha} = t^{1-\alpha}[1 + \mathrm{o}\,(1)]$, Proposition 1 would follow from Lemma 1 together with

$$\begin{aligned} (3.3) \qquad C_1 \exp\left[-\frac{\pi}{2A(1-\alpha)}\, t^{1-\alpha}\right] &\le P_{z_0}\{B^1_{\tau_\alpha} > t\} \\ &\le C_2 \exp\left[-\frac{\pi}{2A(1-\alpha)}\, t^{1-\alpha}\right]. \end{aligned}$$

3.2 Proof of (3.3) by conformal mapping

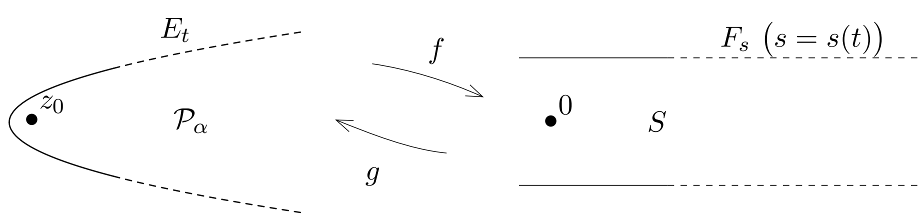
We wish to use the Ahlfors–Warschawski estimates on conformal mappings. Following the standard notation in this area, we introduce the functions $\varphi_+(x) = -\varphi_-(x) = Ax^\alpha$ for $x > 0$ and $\theta(x) = \varphi_+(x) - \varphi_-(x)$, so that $\theta(x) = 2Ax^\alpha$ for $x \ge 0$. Then $\mathcal{P}_\alpha$ has the form

$$\mathcal{P}_\alpha \;=\; \{z = x + iy \colon x > 0 \text{ and } \varphi_-(x) < y < \varphi_+(x)\}.$$

We write $f(z)$ for the conformal mapping of $\mathcal{P}_\alpha$ onto the standard strip

$$S = \{w \colon |\mathrm{Im}\, w| < \pi/2\}$$

for which $f(z_0) = 0$ and $f'(z_0) > 0$. The mapping f is then symmetric in the real axis. We write E_t for $\partial\mathcal{P}_\alpha \cap \{\mathrm{Re}\, z > t\}$ and write F_s for $\partial S \cap \{\mathrm{Re}\, w > s\}$. For $t > 0$, E_t corresponds under the mapping f to F_s, for some s that we denote by $s = s(t)$.



Then, by conformal invariance of harmonic measure, $P_{z_0}\{B^1_{\tau_\alpha} > t\}$ is the harmonic measure at 0 of $F_{s(t)}$ with respect to S, that is

$$(3.4) \qquad P_{z_0}\{B^1_{\tau_\alpha} > t\} = \omega(0, F_{s(t)}; S).$$

Harmonic measure in the strip is easy to estimate, for example by mapping the strip onto the unit disk where harmonic measure at the origin coincides with normalized angular measure on the circle. One may show that

$$\omega(0, F_s; S) = \frac{1}{\pi} \arg\left[e^{2s} - 1 + 2ie^s\right],$$

whence, for s large,

$$\frac{1}{\pi}\, e^{-s} \le \omega(0, F_s; S) \le \frac{4}{\pi}\, e^{-s}. \tag{3.5}$$

Warschawski's estimates [15, Theorem VII] for the real part of the mapping f involve an error term $\int^\infty \theta'(x)^2/\theta(x)\, dx$ for which, in our case,

$$\int_1^\infty \frac{[\,\theta'(x)\,]^2}{\theta(x)}\, dx = A\int_1^\infty \frac{(2\alpha x^{\alpha-1})^2}{2x^\alpha}\, dx = 2\alpha^2 A\int_1^\infty x^{\alpha-2}\, dx < \infty.$$

From [15, Theorem VII] we deduce that, for $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

$$\operatorname{Re} f(z_2) - \operatorname{Re} f(z_1) = \pi\int_{x_1}^{x_2} \frac{dx}{\theta(x)} + \mathrm{o}\,(1),$$

as x_1, $x_2 \to +\infty$, uniformly with respect to y_1 and y_2. As a consequence we obtain that as $t \to \infty$,

$$\operatorname{Re} f(t + iy) = \pi\int_1^t \frac{dx}{\theta(x)} + \mathrm{O}(1) = \frac{\pi}{2A(1-\alpha)}\, t^{1-\alpha} + \mathrm{O}(1).$$

Since this estimate is uniform in y, it follows that

$$s(t) = \frac{\pi}{2A(1-\alpha)}\, t^{1-\alpha} + \mathrm{O}\,(1), \quad \text{as } t \to \infty. \tag{3.6}$$

We will use this estimate on the boundary correspondence under f again in Section 4. Together (3.4), (3.5) and (3.6) yield (3.3). Thus Proposition 1 is proved.

Note It is possible to shortcut the above explicit calculations by using Haliste's estimates for harmonic measure [7, Chapter 1.2], that are themselves based on the Ahlfors-Warschawski approximations. Haliste formulates her estimates in terms of the harmonic measure of a vertical cross cut $\theta_t = [t - iAt^\alpha, t + iAt^\alpha]$ with respect to the truncated domain $\mathcal{P}_\alpha(t) = \{(x, y)\colon 0 < x < t,\, |y| < Ax^\alpha\}$. We set

$$B^{1,*}_{\tau_\alpha} = \sup\{|B^1_t|\colon 0 \le t < \tau_\alpha\}.$$

Then,

$$\omega(z_0, \theta_t; \mathcal{P}_\alpha(t)) = P_{z_0}\{B^{1,*}_{\tau_\alpha} > t\},$$

for which Haliste provides estimates similar to (3.3). A version of Lemma 1 for the maximal functions $B^{1,*}_{\tau_\alpha}$ and $B^*_{\tau_\alpha}$ yields Proposition 1 with $|B_{\tau_\alpha}|$ replaced by $B^*_{\tau_\alpha}$. At this point, it suffices follow the argument in the next section with $B^*_{\tau_\alpha}$ instead of $|B_{\tau_\alpha}|$ and to recall that $|B_{\tau_\alpha}|$ and $B^*_{\tau_\alpha}$ have the same integrability properties (Theorem 1).

3.3 Proof of Theorem 2

As above, we assume for the moment that $z = z_0 = (1, 0)$. First, the limit (1.7) is a direct consequence of Proposition 1. Next we observe that

$$E_{z_0}\left[\exp[b\,|B_{\tau_\alpha}|^a]\right] = 1 + b\int_0^\infty e^{bt} P_{z_0}\{|B_{\tau_\alpha}| > t^{1/a}\}dt. \tag{3.7}$$

Proposition 1 yields that

$$\begin{aligned} C_1 \exp\left[\left(b - \frac{\pi}{2A(1-\alpha)}\right)t\right] &\le e^{bt} P_{z_0}\{|B_{\tau_\alpha}| > t^{1/(1-\alpha)}\} \\ &\le C_2 \exp\left[\left(b - \frac{\pi}{2A(1-\alpha)}\right)t + o(t)\right]. \end{aligned}$$

We suppose first that $b < \pi/[2A(1-\alpha)]$ and set

$$\varepsilon = \frac{\pi}{2A(1-\alpha)} - b.$$

Then ε is positive and we observe that $-\varepsilon t + o(t) \le -\varepsilon t/2$ for sufficiently large t. Hence, for t large,

$$e^{bt} P_{z_0}\{|B_{\tau_\alpha}| > t^{1/(1-\alpha)}\} \le C_2 e^{-\frac{1}{2}\varepsilon t}.$$

It follows from the case $a = 1-\alpha$ of (3.7) that $E_{z_0}\big[\exp[b\,|B_{\tau_\alpha}|^{1-\alpha}]\big] < \infty$ in this case.

In the case $b \ge \pi/[2A(1-\alpha)]$,

$$e^{bt} P_{z_0}\{|B_{\tau_\alpha}| > t^{1/(1-\alpha)}\} \ge C_1 \exp\left[\left(b - \frac{\pi}{2A(1-\alpha)}\right)\ t\right] \ge C_1.$$

This gives $E_{z_0}\big[\exp[b\,|B_{\tau_\alpha}|^{1-\alpha}]\big] = \infty$, for such b.

The cases $0 < a < 1-\alpha$, $b > 0$ and $a > 1-\alpha$, $b > 0$ may be handled similarly, or one may compare the expected value with that of $\exp[b\,|B_{\tau_\alpha}|^{1-\alpha}]$ where, say, $b = \pi/[4A(1-\alpha)]$ and $b = \pi/[A(1-\alpha)]$, respectively.

We now remove the assumption that $z = (1, 0)$. We first deal with the upper bound. We let $z = (x, y)$ and assume that t is very large and certainly much larger than 1. By translation of paths it is clear that $\omega((x_1, y), \theta_t; \mathcal{P}_\alpha(t)) \le \omega((x_2, y), \theta_t; \mathcal{P}_\alpha(t))$ whenever $x_1 \le x_2$. Hence, we may assume that $x \ge 1$. Thus for $t >> x >> 1$ we have, by symmetry,

$$\sup_{z' \in \theta_x} \omega(z', \theta_t; \mathcal{P}_\alpha(t)) = \omega(x, \theta_t; \mathcal{P}_\alpha(t)). \tag{3.8}$$

From this and the more general upper bound of Haliste [7] we have that for $z = (x, y)$ and $t >> x >> 1$,

$$\begin{aligned}\omega(z, \theta(t); \mathcal{P}_\alpha(t)) &\leq C_2 \exp\left[-\pi \int_x^t \frac{du}{\theta(u)}\right] \\ &= C_2 \exp\left[-\frac{\pi}{2A(1-\alpha)}\, t^{1-\alpha} - \frac{\pi}{2A(1-\alpha)}\, x^{1-\alpha}\right]. \qquad (3.9)\end{aligned}$$

These two inequalities, (3.8) and (3.9), give the desired upper bound estimate on $P_z\{B^{1,*}_{\tau_\alpha} > t\}$.

For the lower bound, we may assume by the above argument that $z = (x, y)$ with $0 < x < 1$. As before, we may assume that t is much greater than 1. By a standard Whitney chain argument and the Harnack inequality we have

$$\omega(z, \theta_t; \mathcal{P}_\alpha(t)) \geq C(z)\, \omega(z_0, \theta_t; \mathcal{P}_\alpha(t)), \qquad (3.10)$$

where $C(z)$ is a function of z depending on the distance of z to the boundary of $\mathcal{P}_\alpha$. The general lower bound follows from this and the case of $z = (1, 0)$ which we have already done. $\square$

4 Parabola-shaped regions in $\mathbb{R}^n$

The parabola-shaped regions with which we work have the form

$$\mathcal{P}_\alpha = \{(x, Y) \in \mathbb{R} \times \mathbb{R}^{n-1} \colon x > 0,\ |Y| < Ax^\alpha\},$$

for $0 < \alpha < 1$ and $A > 0$. Our objective is to derive estimates for the distribution function of the exit position of Brownian motion from such regions or, equivalently, for the harmonic measure of the exterior of the ball of center 0 and radius t with respect to such regions. In Section 4.1, we derive an upper bound for the distribution function by means of the Carleman method and in Section 4.2 we introduce a new conformal mapping technique to derive an equally sharp lower bound.

4.1 Carleman method: Upper bound for harmonic measure

It will be more convenient to write the domain $\mathcal{P}_\alpha$ in this section as

$$\mathcal{P}_\alpha = \{(x, Y) \in \mathbb{R} \times \mathbb{R}^{n-1} \colon x > 0,\ Y \in B_{n-1}(0, Ax^\alpha)\},$$

where $B_{n-1}(0,r)$ is the ball in $\mathbb{R}^{n-1}$ centered at 0 and of radius r. For convenience of notation we set $\theta(x) = B_{n-1}(0, Ax^\alpha)$ and refer to $\theta(x)$ as a cross cut of $\mathcal{P}_\alpha$ at x. For t large we set

$$\mathcal{P}_\alpha(t) = \{(x,Y)\colon 0 < x < t,\, Y \in \theta(x)\}.$$

This is the domain $\mathcal{P}_\alpha$ truncated to the right of t. For such a t and any $(x,Y) \in \mathcal{P}_\alpha(t)$, we denote by $\omega\big((x,Y), \theta(t); \mathcal{P}_\alpha(t)\big)$ the harmonic measure of $(t,0) + \theta(t)$ at the point (x,Y) relative to the domain $\mathcal{P}_\alpha(t)$. We wish to estimate $\omega\big((1,0), \theta(t); \mathcal{P}_\alpha(t)\big)$.

Proposition 2. *There exit two constants C_1 and C_2, that depend on n, λ_1, A and α, such that for $t > C_1$,*

$$\omega\big((1,0), \theta(t); \mathcal{P}_\alpha(t)\big) \le C_2\, t^{\alpha(n-1)/2} \ \exp\left[- \frac{\sqrt{\lambda_1}}{A(1-\alpha)}\ t^{1-\alpha} \right]$$

Before we begin the proof of Proposition 2, we show how it leads to an estimate for the distribution of B_{τ_α}. With the notation of Lemma 1 and of the note at the end of Section 3.2,

$$P_{(1,0)}\,\{|B_{\tau_\alpha}| > t\} = P_{(1,0)}\{B^1_{\tau_\alpha} > x(t)\} \le P_{(1,0)}\{B^{1,*}_{\tau_\alpha} > x(t)\}.$$

The distribution function for $B^{1,*}_{\tau_\alpha}$ is the harmonic measure that is estimated in Proposition 2, so that

$$P_{(1,0)}\{B^{1,*}_{\tau_\alpha} > t\} = \omega\big((1,0), \theta(t); \mathcal{P}_\alpha(t)\big).$$

From the estimate (3.2), namely $t - A^2 t^{2\alpha-1} < x(t) < t$, it follows, as in the proof of Lemma 1, that $x(t)^{1-\alpha} = t^{1-\alpha}[1 + \mathrm{o}(1)]$. Together with Proposition 2, this leads to

$$\begin{aligned} P_{(1,0)}\{B^{1,*}_{\tau_\alpha} > x(t)\} \ &\le\ C_2\, x(t)^{\alpha(n-1)/2}\ \exp\left[- \frac{\sqrt{\lambda_1}}{A(1-\alpha)}\ x(t)^{1-\alpha} \right] \\ &\le\ C_2\, t^{\alpha(n-1)/2}\ \exp\left[- \frac{\sqrt{\lambda_1}}{A(1-\alpha)}\ t^{1-\alpha}[1 + \mathrm{o}(1)] \right] \end{aligned}$$

We may absorb the term $C_2\, t^{\alpha(n-1)/2}$ into the $\mathrm{o}(1)$ term in the exponential to deduce an estimate for the distribution function for B_{τ_α} in the following form.

Proposition 3. *Suppose that ϵ is positive. There exists a constant C_1 depending on ϵ, n, λ_1, A and α such that, for $t > C_1$,*

$$P_{(1,0)}\,\{|B_{\tau_\alpha}| > t\} \le \exp\left[- \frac{\sqrt{\lambda_1}}{A(1-\alpha)}\, [1-\epsilon]\, t^{1-\alpha} \right]$$

Proof of Proposition 2. Our estimates follow those of Haliste [7]. We take t to have some large, fixed value and set

$$h(x) = \int_{\theta(x)} \omega^2(x,Y)\,dY, \quad 0 < x < t, \tag{4.1}$$

where for convenience we write $\omega(x,Y)$ for $\omega((x,Y),\theta(t);\mathcal{P}_\alpha(t))$. Differentiating h (see Haliste [7] for the justification of this step), we obtain

$$h'(x) = \int_{\theta(x)} 2\,\omega_x(x,Y)\,\omega(x,Y)\,dY \tag{4.2}$$

and

$$h''(x) = 2\int_{\theta(x)} \omega_{xx}(x,Y)\,\omega(x,Y)\,dY + 2\int_{\theta(x)} |\omega_x(x,Y)|^2\,dY \tag{4.3}$$

We observe that, since $\omega(x,Y)$ is increasing in x for each Y, the derivative $\omega_x(x,Y)$ is non negative and hence $h'(x) \geq 0$. Since $\omega(x,Y)$ is harmonic, we have

$$\omega_{xx}(x,Y) + \Delta_Y\,\omega(x,Y) = 0.$$

Thus,

$$h''(x) = -2\int_{\theta(x)} \Delta_Y\,\omega(x,Y)\,\omega(x,Y)\,dY + 2\int_{\theta(x)} \omega_x(x,Y)^2\,dY. \tag{4.4}$$

Since the harmonic measure vanishes on the lateral boundary of the domain, $\omega(x,Y) = 0$ if $Y \in \partial\theta(x)$, with $0 < x < t$. Thus, integrating by parts, we obtain

$$h''(x) = 2\int_{\theta(x)} |\nabla_Y\,\omega(x,Y)|^2\,dY + 2\int_{\theta(x)} \omega_x(x,Y)^2\,dY. \tag{4.5}$$

Writing $B(0,r)$ for $B_{n-1}(0,r)$, we now recall that for all u that are differentiable on $B(0,r)$ and vanish on $\partial B(0,r)$,

$$\lambda_{B(0,r)} \leq \frac{\int_{B(0,r)} |\nabla u|^2}{\int_{B(0,r)} |u|^2}, \tag{4.6}$$

where $\lambda_{B(0,r)}$ is the first eigenvalue of $B(0,r)$ for the Laplacian. By scaling,

$$\lambda_{B(0,r)} = \frac{1}{r^2}\lambda_1,$$

where λ_1 is the eigenvalue of the unit ball. In our case this gives

$$\lambda_{\theta(x)} = \frac{1}{A^2 x^{2\alpha}}\ \lambda_1. \tag{4.7}$$

From (4.5) and (4.6) we deduce that

$$\begin{aligned} h''(x) &\geq 2\lambda_{\theta(x)} \int_{\theta(x)} \omega(x,Y)^2\, dY + 2\int_{\theta(x)} \omega_x(x,Y)^2\, dY \\ &= 2\lambda_{\theta(x)} h(x) + 2\int_{\theta(x)} \omega_x(x,Y)^2\, dY. \end{aligned} \tag{4.8}$$

On the other hand, by (4.2) and Hölder's inequality,

$$h'(x) \leq 2\left(\int_{\theta(x)} \omega_x(x,Y)^2\, dY\right)^{1/2} \left(\int_{\theta(x)} \omega(x,Y)^2\, dY\right)^{1/2}$$

or

$$h'(x)^2 \leq 4\left(\int_{\theta(x)} \omega_x(x,Y)^2\, dY\right) h(x)$$

or

$$\frac{h'(x)^2}{2h(x)} \leq 2\int_{\theta(x)} \omega_x(x,Y)^2\, dY.$$

This and (4.8) give

$$h''(x) \geq 2\lambda_{\theta(x)} h(x) + \frac{h'(x)^2}{2h(x)}. \tag{4.9}$$

Since $2\sqrt{a}\sqrt{b} \leq a+b$, we conclude that

$$h''(x) \geq 2\sqrt{\lambda_{\theta(x)}}\, h'(x)$$

which, by (4.7), is the same as

$$\frac{h''(x)}{h'(x)} \geq \frac{2\sqrt{\lambda_1}}{A}\ \frac{1}{x^\alpha}.$$

Following Haliste, we consider the function g on the interval $(0,t)$ given by

$$\begin{aligned} g(x) &= \int_0^x \exp\left(2\int_0^s \sqrt{\lambda_{\theta(r)}}\, dr\right) ds \\ &= \int_0^x \exp\left(\frac{2\sqrt{\lambda_1}}{A} \int_0^s \frac{dr}{r^\alpha}\right) ds \\ &= \int_0^x \exp\left(\frac{2\sqrt{\lambda_1}}{A(1-\alpha)}\, s^{1-\alpha}\right) ds. \end{aligned}$$

This function satisfies

$$\begin{aligned} g'(x) &= \exp\left(2\int_0^x \sqrt{\lambda_{\theta(r)}}\,dr\right) \\ g''(x) &= g'(x)\,2\sqrt{\lambda_{\theta(x)}} \\ &= g'(x)\frac{2\sqrt{\lambda_1}}{A}\,\frac{1}{x^\alpha}, \end{aligned}$$

and so

$$\frac{d}{dx}(\log h' - \log g') \geq 0.$$

From this it follows that the function $\frac{h'(x)}{g'(x)}$ is non–decreasing on $(0,t)$. Since $g(0) = h(0) = 0$, the generalized mean value theorem gives that for any $0 < x < t$ there is a $\xi \in (0,x)$ such that $\frac{h(x)}{g(x)} = \frac{h'(\xi)}{g'(\xi)}$. Hence,

$$\frac{h(x)}{g(x)} \leq \frac{h'(x)}{g'(x)}.$$

Since $g'(x) \geq 0$, this shows that $(\frac{h}{g})'(x)$ is nonnegative and hence the function $\frac{h(x)}{g(x)}$ is non–decreasing. Thus,

$$h(x) \leq h(\xi)\,\frac{g(x)}{g(\xi)}, \quad \text{for } 0 < x < \xi < t. \tag{4.10}$$

Setting $\mu(r) = 2\sqrt{\lambda_{\theta(r)}}$,

$$\begin{aligned} g(x) &= \exp\left(\int_0^x \mu(r)\,dr\right)\int_0^x \exp\left(-\int_s^x \mu(r)\,dr\right)ds \\ &= \exp\left(\int_0^x \mu(r)\,dr\right)H(x). \qquad (4.11) \end{aligned}$$

We may estimate $H(x)$ by

$$\begin{aligned} H(x) &= \int_0^x \exp\left(-\frac{2\sqrt{\lambda_1}}{A(1-\alpha)}\left[x^{1-\alpha} - s^{1-\alpha}\right]\right)ds \\ &= \exp\left[-\frac{2\sqrt{\lambda_1}}{A(1-\alpha)}\,x^{1-\alpha}\right]\int_0^x \exp\left[\frac{2\sqrt{\lambda_1}}{A(1-\alpha)}s^{1-\alpha}\right]ds \\ &\leq x. \end{aligned}$$

Therefore,

$$g(x) \leq x\exp\left(\int_0^x \mu(r)\,dr\right), \quad 0 < x < t. \tag{4.12}$$

On the other hand, setting

$$x_0 = x_0(\lambda_1, A, \alpha) = \left[\frac{A(1-\alpha)}{2\sqrt{\lambda_1}} \ln 2 + 1\right]^{1/(1-\alpha)} \qquad \text{and} \qquad K = \frac{2\sqrt{\lambda_1}}{A(1-\alpha)},$$

we find that, for $x \geq x_0$,

$$\begin{aligned}
H(x) &= \exp\big[-K\,x^{1-\alpha}\big] \int_0^x \exp\big[K\,s^{1-\alpha}\big]\,ds \\
&= \exp\big[-K\,x^{1-\alpha}\big] \int_0^{x^{1-\alpha}} \frac{1}{1-\alpha}\, r^{\alpha/(1-\alpha)} \exp\big[K\,r\big]\,dr \\
&\geq \frac{1}{1-\alpha} \exp\big[-K\,x^{1-\alpha}\big] \int_1^{x^{1-\alpha}} \exp\big[K\,r\big]\,dr \\
&= \frac{1}{K(1-\alpha)} \exp\big[-K\,x^{1-\alpha}\big]\big[\exp\big[K\,x^{1-\alpha}\big] - \exp K\big] \\
&= \frac{1}{K(1-\alpha)} \Big[1 - \exp\big[K\,(1 - x^{1-\alpha})\big]\Big] \\
&\geq \frac{1}{2K(1-\alpha)}.
\end{aligned}$$

We have shown that,

$$H(x) \geq \frac{A}{4\sqrt{\lambda_1}} \quad \text{for} \quad x \geq x_0,$$

and this, together with (4.11), gives

$$g(x) \geq \frac{A}{4\sqrt{\lambda_1}} \exp\left(\int_0^x \mu(r)\,dr\right), \quad \text{for} \quad x_0 \leq x < t. \tag{4.13}$$

From (4.10), (4.12) and (4.13) we deduce that,

$$h(x) \leq \frac{4\sqrt{\lambda_1}}{A}\, x\, h(\xi) \exp\left(-\int_x^\xi \mu(r)\,dr\right), \quad \text{for } x_0 \leq x < \xi < t.$$

Taking $x = x_0$ and letting ξ tend to t, we arrive at

$$\begin{aligned}
h(x_0) &\leq \frac{4\sqrt{\lambda_1}}{A}\, x_0 h(t) \exp\left(-\int_{x_0}^t \mu(r)\,dr\right) \\
&= C_1(\lambda_1, A, \alpha)\, h(t) \exp\left[-\frac{2\sqrt{\lambda_1}}{A(1-\alpha)}\, t^{1-\alpha}\right],
\end{aligned}$$

for an appropriate constant C_1. By our definition of h,

$$h(t) \leq \operatorname{vol}\big(B_{n-1}(0, At^\alpha)\big) = \gamma_n A^{n-1} t^{\alpha(n-1)},$$

where γ_n is the volume of the unit ball in $\mathbb{R}^{n-1}$. Thus,

$$h(x_0) \leq C_2(n, \lambda_1, A, \alpha)\, t^{\alpha(n-1)} \exp\left[-\frac{2\sqrt{\lambda_1}}{A(1-\alpha)}\, t^{1-\alpha}\right],$$

for all sufficiently large t. We now choose an r, independent of t, such that the ball $B_n\big((x_0,0), 2r\big)$ is contained in $\mathcal{P}_\alpha$; clearly such an r exists. Then, by the Harnack inequality,

$$\omega(x_0, 0) \leq C_3(n)\, \omega(x, Y) \quad \text{for all } (x, Y) \in B_n\big((x_0, 0), r\big).$$

Squaring and then integrating over the ball $B_{n-1}(0, r)$ leads to

$$\begin{aligned}
\omega^2(x_0, 0) &\leq C_4(n, r) \int_{B_{n-1}(0,r)} \omega^2(x_0, Y)\, dY \\
&\leq C_4(n, r) \int_{\theta(x_0)} \omega^2(x_0, Y)\, dY = C_4(n, r)\, h(x_0).
\end{aligned}$$

From this we finally obtain

$$\omega((x_0, 0), \theta(t); \mathcal{P}_\alpha(t)) \leq C_5(n, \lambda_1, A, \alpha)\, t^{\alpha(n-1)/2} \exp\left[-\frac{\sqrt{\lambda_1}}{A(1-\alpha)}\, t^{1-\alpha}\right],$$

for $t > x_0$. One final application of the Harnack inequality to move from $(x_0, 0)$ to $(1, 0)$, and the proposition is proved. □

4.2 Conformal mapping method: Lower bound

In this section we write $\mathcal{P}_\alpha^n$ for the region in (1.6) to emphasize the dimension. We observe that, because of the cylindrical symmetry of $\mathcal{P}_\alpha^n$ (this region is invariant under rotation about the x-axis) and because of the symmetry of the boundary values of the harmonic measure, the value of the harmonic measure at (x, Y) in $\mathcal{P}_\alpha^n$ depends only on x and on $|Y|$.

We associate with $\mathcal{P}_\alpha^n$ the corresponding domain $\mathcal{P}_\alpha = \mathcal{P}_\alpha^2$ in two dimensions. The technique we develop to obtain lower bounds for the distribution function of the exit position in $\mathcal{P}_\alpha^n$ is the following. We replace Laplace's equation in $\mathcal{P}_\alpha^n$ by the corresponding Bessel-type partial differential equation in $\mathcal{P}_\alpha$ – in the case $n = 2$, this reduces to the Laplacian. Mirroring the

arguments in Section 3.2, we map the parabola-shaped domain $\mathcal{P}_\alpha$ conformally onto the standard strip $S = \{w : |\operatorname{Im} w| < \pi/2\}$ and determine the form the partial differential equation takes in the strip after this change of variable. Adapting the Ahlfors-Warschawski conformal mapping estimates to our purposes, and in some instances refining them, we show that the partial differential equation in the strip is but a small perturbation of the Bessel-type partial differential equation that we started with – it is almost conformally invariant. The solution of the unperturbed Bessel-type p.d.e. in the strip may be easily estimated. We show how to compare the solutions of the perturbed p.d.e. with those of the unperturbed p.d.e., so as to obtain the estimates we require on the original harmonic measure in the parabola-shaped region in $\mathbb{R}^n$. As well as keeping track of how the p.d.e. changes as we change from one domain to another, we also need to keep track of the boundary conditions – but here Warschawski's detailed conformal mapping estimates are exactly what we need.

We break the proof into a number of subsections and lemmas.

4.2.1 From a parabola-shaped region in $\mathbb{R}^n$ to a planar strip

To begin with we compute how the Laplace operator changes as we drop down from n dimensions to 2 dimensions.

Lemma 2. *Suppose that $H(x, Y)$ is a C^2-function on $\mathcal{P}_\alpha^n$ that is invariant under rotation about the x-axis, so that H depends only on x and on $|Y|$. We associate with H a function $h(z)$ in the half parabola-shaped planar domain*

$$\mathcal{P}_\alpha^+ \;=\; \{z = x + iy \colon x > 0 \text{ and } 0 < y < Ax^\alpha\},$$

defined by

$$h(x + iy) = H(x, Y) \quad \textit{whenever} \quad |Y| = y.$$

Then

$$\Delta H(x, Y) = \Delta h(x + iy) + (n - 2)\frac{h_y(x + iy)}{y}. \tag{4.14}$$

Proof. In this proof, we denote a point in $\mathcal{P}_\alpha^n$ by $(x_1, x_2, \ldots, x_n)$. With this notation,

$$H(x_1, x_2, \ldots, x_n) = h\left(x_1, \sqrt{x_2^2 + \ldots + x_n^2}\right) = h(x, y)$$

with $x = x_1$ and $y = \sqrt{x_2^2 + \ldots + x_n^2}$. Then,

$$
\begin{aligned}
\Delta H &= \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} h(x_1, \sqrt{x_2^2 + \ldots + x_n^2}) \\
&= \frac{\partial^2 h}{\partial x^2} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \left[\frac{\partial h}{\partial y} \frac{x_i}{\sqrt{x_2^2 + \ldots + x_n^2}} \right] \\
&= \frac{\partial^2 h}{\partial x^2} + \sum_{i=2}^{n} \left[\frac{\partial h}{\partial y} \frac{(x_2^2 + \ldots + x_n^2) - x_i^2}{(x_2^2 + \ldots + x_n^2)^{3/2}} + \frac{\partial^2 h}{\partial y^2} \frac{x_i^2}{x_2^2 + \ldots + x_n^2} \right] \\
&= \frac{\partial^2 h}{\partial x^2} + \frac{\partial h}{\partial y} \left[\frac{n-1}{\sqrt{x_2^2 + \ldots + x_n^2}} - \frac{1}{\sqrt{x_2^2 + \ldots + x_n^2}} \right] + \frac{\partial^2 h}{\partial y^2} \\
&= \Delta h(z) + (n-2) \frac{h_y(z)}{y}.
\end{aligned}
$$

$\square$

From now on we may work in two dimensions and have conformal mapping at our disposal, at the expense of having to deal with the more complicated Bessel-type differential operator appearing in (4.14), rather than the Laplacian. The complication arises because this operator is not conformally invariant.

As in Section 3.2, we denote by $w = f(z)$ the conformal mapping from the domain $\mathcal{P}_\alpha$ onto the standard strip S, for which $f(1) = 0$ and $f'(1) > 0$. Since f is real on the real axis, the upper half $\mathcal{P}_\alpha^+$ of the parabola-shaped domain $\mathcal{P}_\alpha$ is mapped to the upper half S^+ of the strip, specifically, $S^+ = \{w \colon 0 < \operatorname{Im} w < \pi/2\}$.

We denote the inverse mapping of $f(z)$ by $g(w)$. We associate a function $k(w)$ in S^+ with a function $h(z)$ in $\mathcal{P}_\alpha^+$ according to

$$k(w) = h(g(w)), \quad \text{for } w \in S^+. \tag{4.15}$$

Then $h(z) = k(f(z))$, for $z \in \mathcal{P}_\alpha^+$. In the next lemma, we compute how the differential operator on the right of (4.14) changes under this change of variables.

Lemma 3. *Suppose that $h(z)$ is a C^2-function in the domain $\mathcal{P}_\alpha^+$ and that k is defined in the strip S^+ by (4.15). Then, with $g(w) = z$,*

$$\Delta h(z) + (n-2) \frac{h_y(z)}{y} = \frac{\Delta k(w)}{|g'(w)|^2} - 2(n-2) \frac{\operatorname{Im}\,[k_w(w) \,/\, g'(w)]}{\operatorname{Im}\,[g(w)]}. \tag{4.16}$$

Proof. We recall that $h(z) = k(f(z))$, and use the formulae for change of variable in the complex partial derivatives $\partial/\partial z$ and $\partial/\partial\overline{z}$, as explained, for example, in Krantz [8, Section 1.2]. First,

$$\Delta h(z) = (\Delta k)(f(z))\,|f'(z)|^2, \qquad z \in \mathcal{P}_\alpha.$$

In general,

$$\frac{\partial}{\partial(\operatorname{Im} z)} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right),$$

whence

$$\begin{aligned} h_y(z) &= i\left(\frac{\partial h}{\partial z} - \frac{\partial h}{\partial \overline{z}}\right) \\ &= i\left(\frac{\partial}{\partial z}[k(f(z))] - \frac{\partial}{\partial \overline{z}}[k(f(z))]\right) \\ &= i\left(\frac{\partial k}{\partial w}(f(z))\,f'(z) - \frac{\partial k}{\partial \overline{w}}(f(z))\,\overline{f'(z)}\right). \end{aligned}$$

Since $k(w)$ is real-valued, $k_{\overline{w}} = \overline{k_w}$. We therefore obtain,

$$h_y(z) = -2\operatorname{Im}\left[k_w\big(f(z)\big)\,f'(z)\right].$$

Thus,

$$\begin{aligned} &\Delta h(z) + (n-2)\frac{h_y(z)}{y} \\ &\qquad = (\Delta k)(f(z))\,|f'(z)|^2 - 2(n-2)\frac{\operatorname{Im}\left[k_w\big(f(z)\big)\,f'(z)\right]}{\operatorname{Im}\,[g(w)]}. \end{aligned}$$

On substituting w for $f(z)$ and $1/g'(w)$ for $f'(z)$, we obtain (4.16). $\square$

4.2.2 Asymptotic estimates for the conformal mapping g

The success of the transformations introduced in the previous subsection depends on being able to simplify the expression on the right hand side of (4.16), which is essentially an expression for the Laplacian in the domain $\mathcal{P}_\alpha^n$ in $\mathbb{R}^n$ transformed to the strip S^+ in the plane. This is achieved using modifications of results of Warschawski [15], which give asymptotic expressions for the conformal mapping g and for its derivative. For some of these we draw on [4]. We begin with an estimate for the imaginary part of the mapping g. In this context it is helpful to keep in mind that estimates for the

real part of the mapping g, that is for the rate of growth of g, are generally more difficult.

We will adopt the more general situation, as considered by Warschawski, of a conformal mapping f of a domain of the form

$$D = \{z : |\operatorname{Im} z| < \phi(\operatorname{Re} z)\},$$

where $\phi(x)$ is continuous on the real line, onto the strip S. In our case, $\phi(x) = Ax^\alpha$, for x positive. Warschawski's domains are not necessarily symmetric, but the symmetric case is sufficiently general for our purposes here. Warschawski writes $\theta(x) = 2\phi(x)$ for the width of the domain D at x. We assume that D has *boundary inclination 0 at* $x = \infty$ in that

$$\phi(x_2) - \phi(x_1) = \mathrm{o}\,(x_2 - x_1) \quad \text{as } x_1,\, x_2 \to \infty.$$

This is the case if $\phi(x)$ is continuously differentiable and $\phi'(x) \to 0$ as $x \to \infty$, and includes our parabola-shaped domains $\mathcal{P}_\alpha$.

We take the conformal mapping f of D onto S for which f is real and f' is positive on the real axis. We denote the inverse mapping of f by g, in agreement with earlier notation. We begin with an estimate for the derivative of g.

Lemma A. Warschawski, [15, Theorem X(ii)] *For each ρ with $0 < \rho < 1$, the conformal mapping g of the strip S onto D, for which g is real and g' is positive on the real axis, satisfies*

$$g'(w) = \left[\frac{1}{\pi} + \mathrm{o}(1)\right] \theta(\operatorname{Re} g(w)), \tag{4.17}$$

uniformly as $\operatorname{Re} w \to \infty$ *in the sub strip* $S_\rho = \{w : |\operatorname{Im} w| < \rho\pi/2\}$.

In [15, Theorem X(iii)], Warschawski obtains an asymptotic expression for $\operatorname{Im} g(w)$ as $\operatorname{Re} w \to \infty$. Taking advantage of the symmetry of the domain D (so that $\operatorname{Im} g(w) = 0$ when $\operatorname{Im} w = 0$) and adapting Warschawski's proof, we prove

Lemma 4. *The conformal mapping g of the strip S onto D, for which g is real and g' is positive on the real axis, satisfies*

$$\operatorname{Im} g(w) = \left[\frac{1}{\pi} + \mathrm{o}\,(1)\right] \theta(g(\operatorname{Re} w)) \operatorname{Im} w, \tag{4.18}$$

as $\operatorname{Re} w \to \infty$ *with* $w \in S$.

Proof. By symmetry of the mapping g, it is enough to prove (4.18) in the case of $\operatorname{Im} w$ positive. First we show that (4.18) holds in each sub strip S_ρ, $(0 < \rho < 1)$. By Theorem II(a) in [15],

$$\lim_{u\to\infty} \arg g'(u+iv) = 0, \tag{4.19}$$

uniformly in v, $|v| < \pi/2$. Combining (4.19) with Lemma A, we find that

$$\operatorname{Re} g'(w) = \left[\frac{1}{\pi} + \mathrm{o}\,(1)\right] \theta(\operatorname{Re} g(w)),$$

for $w \in S_\rho$. Then, for $w = u + iv_0 \in S_\rho$,

$$\begin{aligned}
\operatorname{Im} g(u+iv_0) &= \operatorname{Im} g(u+iv_0) - \operatorname{Im} g(u) \\
&= v_0 \left(\frac{\partial}{\partial v}\operatorname{Im} g(u+iv)\right)\Big|_{v=v_1} \qquad [v_1 \in (0, v_0)] \\
&= v_0 \operatorname{Re} g'(u+iv_1) \\
&= v_0 \left[\frac{1}{\pi} + \mathrm{o}\,(1)\right] \theta(\operatorname{Re} g(u+iv_1)).
\end{aligned}$$

We assert that, uniformly in v, $|v| < \pi/2$,

$$\theta(\operatorname{Re} g(u+iv)) = \theta(g(u)) + \mathrm{o}(1), \tag{4.20}$$

(which is why we need not distinguish between $\theta(g(\operatorname{Re} w))$ and $\theta(\operatorname{Re} g(w))$, up to $\mathrm{o}\,(1)$). This then gives the stated expression for $\operatorname{Im} g(u+iv_0)$, for $|v_0| < \rho\pi/2$. To prove (4.20) we note that, as at the bottom of Page 290 of [15] and as a consequence of the assumption that D has boundary inclination 0 at $x = \infty$,

$$\operatorname{Re} g(u+iv) = g(u) + \mathrm{o}(1), \quad \text{as } u \to \infty.$$

On using yet again the assumption that D has boundary inclination 0 at $x = \infty$, (4.20) follows.

Finally, we need to show that (4.18) holds uniformly in v. Given ϵ small and positive, we take $\rho = 1 - \epsilon$, so that (4.18) holds for $w = u + iv$, with $|v| \le (1-\epsilon)\pi/2$. In particular,

$$\begin{aligned}
\operatorname{Im} g\left(u + i\frac{\pi}{2}(1-\epsilon)\right) &= \left[\frac{1}{\pi} + \mathrm{o}\,(1)\right] \theta(g(u))\frac{\pi}{2}(1-\epsilon) \\
&= (1-\epsilon)\frac{\theta(g(u))}{2} + \mathrm{o}\,(\theta(g(u))),
\end{aligned}$$

as $u \to \infty$. Thus, using (4.20), the image of the sub strip S_ρ is a region of the form

$$\left\{z : |\operatorname{Im} z| \le (1-\epsilon)\frac{\theta(\operatorname{Re} z)}{2} + \mathrm{o}\,(\theta(\operatorname{Re} z))\right\}.$$

Since $g(w)$ lies outside this region if $v > (1-\epsilon)\pi/2$, we find that

$$\text{(4.21)} \qquad \begin{aligned} \operatorname{Im} g(w) &\geq (1-\epsilon)\frac{\theta(\operatorname{Re} g(w))}{2} + \mathrm{o}\left(\theta(\operatorname{Re} g(w))\right) \\ &= (1-\epsilon)\frac{\theta(g(u))}{2} + \mathrm{o}\left(\theta(g(u))\right). \end{aligned}$$

On the other hand, if $v > (1-\epsilon)\pi/2$ then, simply because $g(w)$ lies in D,

$$\text{(4.22)} \qquad \operatorname{Im} g(w) \leq \frac{\theta(\operatorname{Re} g(w))}{2} = \frac{\theta(g(u))}{2} + \mathrm{o}(1).$$

Together (4.21) and (4.22) imply that, for u sufficiently large and $v > (1-\epsilon)\pi/2$,

$$\left|\operatorname{Im} g(w) - \frac{\theta(g(u))}{\pi} v\right| \leq 2\epsilon\, \theta(g(u)).$$

It follows that (4.18) holds uniformly in v, $-\pi/2 < v < \pi/2$, with $\mathrm{o}(1)$ replaced by $C\epsilon$. Since ϵ may be as small as we please, the proof of (4.18) is complete. $\square$

We wish to remove the restriction in Warschawski's Lemma A that w lies in a fixed sub strip of the standard strip S, at least when D is a parabola-shaped domain. In this situation,

$$\theta(x) = 2Ax^{\alpha}, \quad \text{for } x > 0.$$

Proposition 4. *We set g to be the conformal mapping of the strip S onto $\mathcal{P}_\alpha$ for which g is real and g' is positive on the real axis. Then the following estimate for the derivative of g holds:*

$$\text{(4.23)} \qquad g'(w) = \left[\frac{1}{\pi} + \mathrm{o}(1)\right] \theta(g(\operatorname{Re} w)) = 2A\left[\frac{1}{\pi} + \mathrm{o}(1)\right] g(u)^{\alpha},$$

as u, the real part of w, tends to ∞, uniformly in $\operatorname{Im} w$.

Proposition 4 is part of the main result in [4], in which it is shown that the function

$$h(z) = -\exp\left[\frac{\pi}{2(1-\alpha)}\left[1-(1+Bz)^{1-\alpha}\right]\right], \qquad B = A^{-1/(1-\alpha)},$$

is univalent in $\mathcal{P}_\alpha$ and maps $\mathcal{P}_\alpha$ onto the interior D of a Dini-smooth curve C lying inside the unit circle $|z| = 1$, except for $z = -1$. A smooth Jordan

curve C is said to be Dini-smooth if there is an increasing function $\omega(x)$, that satisfies the Dini-condition

$$\int_0^1 \frac{\omega(x)}{x}\,dx < \infty,$$

for which the angle $\beta(s)$ of the tangent to C, considered as a function of arclength, satisfies

$$|\beta(s_2) - \beta(s_1)| < \omega(s_2 - s_1).$$

The proof in [4] that $h(\mathcal{P}_\alpha)$ is indeed bounded by a Dini-smooth curve, that lies inside the unit circle except for $z = -1$, is quite involved. Proposition 4 follows from this in a relatively straightforward manner. We repeat a version of the argument here for the reader's convenience.

We first note that

$$h'(z) \;=\; -\frac{\pi}{2}\,B\,(1+Bz)^{-\alpha}\,h(z), \quad z \in \mathcal{P}_\alpha. \tag{4.24}$$

We consider the mapping

$$\Psi(z) = h\left[g\left(\log\frac{1+z}{1-z}\right)\right], \quad z \in \Delta.$$

The unit disk Δ is first mapped onto the strip S, which is mapped by g onto the parabola-shaped domain $\mathcal{P}_\alpha$ and, finally, this is mapped by h onto the inside of a Dini-smooth curve C. As explained in [4], we are now in a position to apply [13, Theorem 10.2] and may deduce that Ψ' has a continuous, non-zero extension to the closure of the unit disk. In particular, $\Psi'(1) \neq 0$, and we note that $\Psi(1) = 0$. We can derive information on the behaviour of the derivative of g from information on the derivative of Ψ. We write $w(z) = \log[(1+z)/(1-z)]$, for $z \in \Delta$. Using the expression (4.24) for the derivative of h,

$$\begin{aligned}
\Psi'(z) &= h'\big(g(w(z))\big)\,g'\big(w(z)\big)\,w'(z)\\
&= -\frac{\pi}{2}\,B\,\big[1+Bg(w(z))\big]^{-\alpha}\psi(z)\,g'\big(w(z)\big)\frac{2}{1-z^2}\\
&= -\pi B\,\frac{1}{1+z}\,\frac{g'\big(w(z)\big)}{\big[1+Bg(w(z))\big]^{\alpha}}\,\frac{\Psi(z)}{1-z}.
\end{aligned}$$

We let $z \to 1$ from within the unit disk. Then

$$\frac{\Psi(z)}{1-z} \;\to\; -\Psi'(1),$$

which is non-zero. Hence,

$$\frac{g'\big(w(z)\big)}{\big[1+Bg(w(z))\big]^{\alpha}} \;\to\; \frac{2}{\pi B}, \quad \text{as } z \to 1,\; z \in \Delta.$$

It is not difficult to see that $(\operatorname{Re} z)/(1+Bz) \to B^{-1}$ as $\operatorname{Re} z \to \infty$ with $z \in \mathcal{P}_\alpha$. Substituting $z = g(w)$ in this limit yields

$$\frac{\operatorname{Re} g(w)}{1+Bg(w)} \to \frac{1}{B} \quad \text{as } \operatorname{Re} w \to \infty \text{ with } w \in S.$$

Since the unrestricted limit as $z \to 1$ within the unit disk corresponds to the unrestricted limit as $\operatorname{Re} w \to \infty$ within the strip S, it follows from the previous two estimates that

$$\frac{g'(w)}{[\operatorname{Re} g(w)]^{\alpha}} \;\to\; \frac{2}{\pi B}\, B^{\alpha} = \frac{2A}{\pi}, \quad \text{as } \operatorname{Re} w \to \infty,\; w \in S,$$

which is (4.23).

4.2.3 Asymptotic form of the differential operator

Armed with the asymptotics for $\operatorname{Im} g(w)$ in Lemma 4 and those for $g'(w)$ in Proposition 4, we are now ready to derive the asymptotics for the differential operator

$$\frac{\Delta k(w)}{|g'(w)|^2} - 2(n-2)\frac{\operatorname{Im}\,[k_w(w)\,/\,g'(w)]}{\operatorname{Im}\,[g(w)]}, \tag{4.25}$$

which acts on C^2-functions defined in the strip S and arises in Lemma 3. We keep in mind that, for a real-valued function $k(w) = k(u+iv)$,

$$k_v(w) = -2\operatorname{Im}\big(k_w(w)\big).$$

First we compute, using Proposition 4,

$$\begin{aligned} -2\operatorname{Im}\left[\frac{k_w(w)}{g'(w)}\right] &= -2\operatorname{Im}\left[\frac{k_w(w)}{\big[1/\pi + \mathrm{o}\,(1)\big]\theta\big(g(u)\big)}\right] \\ &= -2\,\frac{\pi}{\theta\big(g(u)\big)}\operatorname{Im}\,\big[k_w(w)\big(1+\mathrm{o}\,(1)\big)\big] \\ &= -2\,\frac{\pi}{\theta\big(g(u)\big)}\left[-\frac{1}{2}k_v(w) + \operatorname{Im}\big[\mathrm{o}\,(1)k_w(w)\big]\right] \end{aligned}$$

$$
\begin{aligned}
&= -2\,\frac{\pi}{\theta\big(g(u)\big)}\left[-\frac{1}{2}k_v(w) + \mathrm{o}\,(1)k_v(w)\right] \\
&= \frac{\pi}{\theta\big(g(u)\big)}\,k_v(w)\big[1 + \mathrm{o}\,(1)\big].
\end{aligned}
$$

Using Lemma 4 to estimate the imaginary part of $g(w)$, we find that

$$
\begin{aligned}
-2\,\frac{\mathrm{Im}\,[k_w(w)\,/\,g'(w)]}{\mathrm{Im}\,[g(w)]} &= \frac{\pi}{\theta\big(g(u)\big)}\,\frac{k_v(w)[1+\mathrm{o}(1)]}{[1/\pi + \mathrm{o}(1)]\theta\big(g(u)\big)\,v} \\
&= \frac{\pi^2}{\theta^2\big(g(u)\big)}\,\frac{k_v(w)}{v}\big[1+\mathrm{o}(1)\big].
\end{aligned}
$$

Similarly, and again using Proposition 4,

$$
\frac{1}{|g'(w)|^2} = \frac{\pi^2}{\theta^2\big(g(u)\big)}\big[1+\mathrm{o}(1)\big].
$$

In summary, the differential operator in (4.25) becomes

$$
\frac{\pi^2}{\theta^2\big(g(u)\big)}\left[\big[1+\mathrm{o}(1)\big]\Delta k(w) + (n-2)\big[1+\mathrm{o}(1)\big]\frac{k_v(w)}{v}\right]. \tag{4.26}
$$

Summary to date: The work in the foregoing sections has been leading up to the following. Suppose that H is harmonic in the domain $\mathcal{P}_\alpha^n$ in $\mathbb{R}^n$ and that $H(x,Y)$ is rotationally symmetric about the x-axis. Suppose that the function h in the planar domain $\mathcal{P}_\alpha^+$ is constructed from H according to $h(x+iy) = H(x,Y)$, with $|Y| = y$. Then, by Lemma 2, h satisfies

$$
\Delta h(z) + (n-2)\frac{h_y(z)}{y} = 0, \qquad z \in \mathcal{P}_\alpha^+.
$$

From $h(z)$ we construct the function $k(w)$ in the strip S^+ according to

$$
k(w) = h\big(g(w)\big), \qquad w \in S^+,
$$

where g is a symmetric conformal mapping of the strip S onto the parabola-shaped domain $\mathcal{P}_\alpha$. The partial differential equation satisfied by k is given in the next proposition, which follows directly from (4.26) and Lemma 3.

Proposition 5. *There is a function $\epsilon(w)$ in the strip S, with the properties that*

(i) $\epsilon(w) \to 0$ as $u \to \infty$, uniformly in v,

(ii) whenever the function $k(w)$ arises from a rotationally symmetric harmonic function H in $\mathcal{P}_\alpha^n$, as described above, then k satisfies the partial differential equation

$$\Delta k(w) + [n + \epsilon(w) - 2]\frac{k_v(w)}{v} = 0, \quad w \in S^+. \tag{4.27}$$

Remark 1. *If the harmonic function H with which we began had lived in a cylinder in $\mathbb{R}^n$ (of radius $\pi/2$ and with axis along the x-axis), then the associated function h would have the standard strip S as its domain of definition. The mapping g would be the identity mapping and so k would simply satisfy $\Delta k(w) + (n-2)k_v(w)/v = 0$ in this case. Proposition 5 may be thought of as asserting that k behaves asymptotically as if it derived from a cylindrical domain. One may also interpret Proposition 5 as asserting that while the differential operator $\Delta h(z) + (n-2)h_y(z)/y$ is not conformally invariant in the same way that the Laplacian is, it is asymptotically conformally invariant. The conformal invariance of the Laplacian was used in (3.4) in Section 3.2, and Proposition 5 is essentially an extension of this to higher dimensions.*

4.2.4 Sub solutions and a maximum principle

We need to determine the boundary conditions satisfied by a function k that is constructed, as in Proposition 5, from the rotationally symmetric harmonic function

$$H(x, Y) = P_{(x,Y)}\{|B_{\tau_\alpha}| > t\},$$

in the region $\mathcal{P}_\alpha^n$ in $\mathbb{R}^n$. Thus H is the harmonic measure of the exterior of the ball of radius t w.r.t. $\mathcal{P}_\alpha^n$. The gradient of H w.r.t. Y vanishes when $x = 0$ because of the rotational symmetry. This translates into the boundary condition $h_y(x, 0) = 0$ for the associated function h in the planar domain $\mathcal{P}_\alpha^+$ and, in turn, into the boundary condition $k_v(u, 0) = 0$ for the function $k(u, v)$ in the strip S^+:

$$k_v(u) = 0, \qquad -\infty < u < \infty. \tag{4.28}$$

The boundary values of H lead to the condition on the boundary of $\mathcal{P}_\alpha^+$ that $h(x, Ax^\alpha) = 1$ if $|(x, Ax^\alpha)| > t$ and $h(x, Ax^\alpha) = 0$ if $|(x, Ax^\alpha)| < t$. Under the conformal mapping f of the domain $\mathcal{P}_\alpha$ onto the strip S, this becomes the following boundary condition for k:

$$k(u + i\pi/2) = 0, \quad -\infty < u < s; \tag{4.29}$$

$$k(u + i\pi/2) = 1, \quad s < u < \infty. \tag{4.30}$$

Here s depends on t, as specified in (3.6). We note that the point $(1, 0)$ in $\mathcal{P}^n_\alpha$ corresponds to the point 0 on the boundary of the strip S^+.

Let us therefore suppose that k is a solution of the p.d.e.

$$\Delta k(w) + [n + \epsilon(w) - 2]\frac{k_v(w)}{v} = 0, \qquad w \in S^+, \tag{4.31}$$

where $\epsilon(w) \to 0$ as $u \to \infty$, uniformly in v, with the boundary conditions (4.28), (4.29) and (4.30). We will show in the next section that $k(0)$ decays at a slower exponential rate as $s \to \infty$ than solutions of

$$\Delta k(w) + [n + \delta - 2]\frac{k_v(w)}{v} = 0, \qquad w \in S^+, \tag{4.32}$$

when δ is positive, the boundary conditions being the same as those satisfied by k.

This comparison between the solutions of (4.31) and (4.32) has a heuristic interpretation that may be helpful to keep in mind. In the limiting case $\delta = 0$, the differential equation (4.32) becomes $\Delta k(w) + (n-2)k_v(w)/v = 0$, the solutions of which, with the above boundary conditions, may be thought of as deriving from harmonic measure in a cylinder of radius $\pi/2$ in $\mathbb{R}^n$. The solutions of (4.32) may then be thought of as corresponding to harmonic measure in such a cylinder in a slightly higher 'dimension' when δ is positive. Thus, our results will show that the distribution function of the exit position of Brownian motion from a parabola-shaped region in $\mathbb{R}^n$ decays like the distribution function of the exit position from a cylinder in $\mathbb{R}^n$, but with a time change that is given explicitly by (3.6).

It is natural to consider solutions of (4.32) in the half strip

$$S^+_s = S^+ \cap \{u < s\} = \{w = u + iv : u < s \text{ and } 0 < v < \pi/2\}.$$

In fact, by symmetry, a solution of the p.d.e. (4.32) in S^+ that satisfies the boundary conditions (4.28), (4.29) and (4.30) will take the constant value $1/2$ on the vertical cross cut $u = s$. This boundary condition can be satisfied by using separation of variables to solve (4.32) in the half strip S^+_s and then taking a series expansion. The rate of decay of the solution at 0 as s becomes large is then determined by the first term in this Bessel series. This is the term that is therefore of interest to us. For each m, we write $J_m(v)$ for the

Bessel function of order m, we write j_m for its smallest positive zero and we set

$$\hat{J}_m(v) = v^{-m} J_m(v).$$

The first term in the Bessel series for a solution of (4.32) in S_s^+ is (a constant times)

$$\phi_\delta(w) = e^{2j_m(u-s)/\pi}\, \hat{J}_m\left(\frac{2j_m}{\pi}\, v\right), \quad \text{where } m = \frac{1}{2}(n+\delta-3). \tag{4.33}$$

Since $\hat{J}_m$ satisfies the differential equation (see [16, Section 17.22], for example),

$$\hat{J}_m''(v) + [2m+1]\frac{\hat{J}_m'(v)}{v} + \hat{J}_m(v) = 0,$$

$\phi_\delta(w)$ satisfies the p.d.e. (4.32) in S_s^+, as well as the boundary conditions (4.28) and (4.29). On the vertical side of the half strip S_s^+, its values are simply $\hat{J}_m\big(2j_m v/\pi\big)$. One needs to take the entire series to have a solution which equals $1/2$ on the vertical cross cut $u = s$.

We write L for the operator

$$L[f] = \Delta f + \big[n + \epsilon(w) - 2\big]\,\frac{f_v}{v}. \tag{4.34}$$

Then $k(w)$ is a solution of $L[k] = 0$ in the half strip S_s^+ with the boundary conditions (4.28) and (4.29). In order to compare k to solutions of (4.32), we construct sub solutions for L in S_s^+, and then use a maximum principle. Of course, all our estimates need to be uniform in s. We show how to obtain the sub solutions that we need in the next lemma.

Lemma 5. *We suppose that, for a fixed positive δ, the number u_δ is chosen so large that $2|\epsilon(w)| < \delta$ for $u > u_\delta$. We suppose that a function k_δ is defined in the rectangle $R_s = S_s^+ \cap \{u > u_\delta\}$ and satisfies the partial differential equation (4.32) there. Suppose further that $\partial k_\delta/\partial v$ is negative in R_s. Then, $L[k_\delta] \geq 0$ in R_s. In particular, $L[\phi_\delta] \geq 0$.*

Proof. With k_δ as in the statement of the lemma,

$$\begin{aligned} L[k_\delta] &= \Delta k_\delta + \big[n + \epsilon(w) - 2\big]\frac{1}{v}\frac{\partial k_\delta}{\partial v} \\ &= \Delta k_\delta + \big[n + \delta - 2\big]\frac{1}{v}\frac{\partial k_\delta}{\partial v} + \big[\epsilon(w) - \delta\big]\frac{1}{v}\frac{\partial k_\delta}{\partial v} \\ &= \big[\epsilon(w) - \delta\big]\frac{1}{v}\frac{\partial k_\delta}{\partial v} \end{aligned}$$

Since $2|\epsilon(w)| < \delta$ for $u > u_\delta$, it follows that $\epsilon(w) - \delta$ has the same sign as $-\delta$ in the rectangle R_s. Since $\partial k_\delta / \partial v$ is negative in R_s, we deduce that $L[k_\delta]$ has the same sign as δ in R_s.

The statement about ϕ_δ now follows from the facts that ϕ_δ satisfies (4.32) and that $\hat{J}_m$ is decreasing on the interval $(0, j_m)$.

□

The other ingredient we need is an appropriate form of the maximum principle. While the version presented here is most probably not new, we have been unable to find it in the literature. Consequently, we outline the proof for completeness.

Lemma 6. *If f is a sub solution of L which is C^2 in the closure of the rectangle R_s (the second derivatives are continuous up to the boundary) and which is non positive on the top, left and right parts of the boundary, then $f(u,v) \leq 0$ for any $(u,v) \in R_s$.*

Proof. Let $Z_t = (X_t, Y_t)$ be the diffusion associated with the operator L. Then $Y_t > 0$ almost surely for all t. This is true in the case $\epsilon(w) = 0$, since Z_t is then a Bessel process and as such it never hits zero (see [11, Chapter XI]). If ϵ is not zero, we still assume that $-\delta \leq \epsilon(x,y) \leq \delta$ in the rectangle. It follows by a stochastic comparison theorem argument (as in the classical Ikeda–Watanabe theorem, [12]) that $Y_t > 0$ almost surely for all $t > 0$. Now, let τ be the first time that Z_t hits the boundary of the rectangle with the diffusion starting at $z_0 = (x_0, y_0) \in R_s$. This time is finite almost surely. Of course, by the above, Z_τ belongs only to the three sides of the rectangle with probability 1. Applying Itô's Lemma,

$$f(Z_{t \min \tau}) - f(z_0) = M_t + A_t \tag{4.35}$$

where M_t is a martingale and $A_t = \int_0^t L[f](Z_s)ds$. Since f is a sub solution of L, we have $L[f] \geq 0$. Taking expectations of both sides of (4.35), we conclude that

$$f(z_0) \leq E_{z_0}\left(f(Z_{t \min \tau})\right).$$

We now let $t \to \infty$. Since f is bounded in the closure of the rectangle,

$$f(z_0) \leq E_{z_0}(f(Z_\tau)),$$

which proves $f(z_0) \leq 0$.

□

4.2.5 Rate of exponential decay

We now have all the ingredients necessary to prove the following estimate of k.

Proposition 6. *Suppose that the function $\epsilon(w)$, $w \in S$, satisfies $\epsilon(w) \to 0$ as $u \to \infty$, uniformly in v. Suppose that $k(w)$ is the solution of (4.27) in S^+ with the boundary conditions (4.28), (4.29) and (4.30), so that k derives from harmonic measure in $\mathcal{P}_\alpha^n$ as in Section 4.2.3. Then, given ϵ positive,*

$$k(0) \geq \exp\left[-\left(\frac{2j_m}{\pi} + \epsilon\right) s\right],$$

for all sufficiently large s, where $m = (n-3)/2$.

Proof. The function k is bounded by 1 on the vertical side $u = s$ of R_s (since it is but a certain harmonic measure in the parabola-shaped region $\mathcal{P}_\alpha^n$ in disguise). More precisely, $k(s, v) \to 1/2$ as $s \to \infty$, uniformly for v in $(0, \pi/2)$. In fact, the harmonic measure of $\partial\mathcal{P}_\alpha^n \cap \{x > t\}$ w.r.t. the parabola-shaped region $\mathcal{P}_\alpha^n$ approaches $1/2$ uniformly on the cross section $\{(t, Y) : Y \in \mathbb{R}^{n-1},\ |Y| < At^\alpha\}$. Hence, for all sufficiently large s,

$$\frac{1}{4} \leq k(s, v) \leq 1, \quad \text{for } 0 < v < \pi/2. \tag{4.36}$$

Given ϵ positive, we set $m_1 = \frac{1}{2}(n + \delta - 3)$ and choose δ positive, but so small that $j_{m_1} \leq j_m + \pi\epsilon/4$. This is possible since the first positive zero of the Bessel function depends continuously on the order of the Bessel function and increases with this order. We suppose that u_δ is as in Lemma 5 and that $s > u_\delta$. Direct comparison of k with the function ϕ_δ of (4.33) doesn't quite work, as ϕ_δ is positive on the side $u = u_\delta$ of R_s while we do not know k there. We consider the positive function

$$k_\delta(w) = \left[e^{2j_{m_1}(u-s)/\pi} - e^{2j_{m_1}[(u_\delta - s)-(u-u_\delta)]/\pi}\right] \hat{J}_{m_1}\left(\frac{2j_{m_1}}{\pi}\, v\right), \quad w \in R_s,$$

in which the second exponential term compensates for the positive values of ϕ_δ on the side $u = u_\delta$. This function is a solution of (4.32) in R_s and, moreover, $\partial k_\delta/\partial v$ is negative in R_s since $\hat{J}_{m_1}$ is decreasing on $(0, j_{m_1})$. It follows from Lemma 5 that $L[k_\delta] \geq 0$. The function k_δ satisfies zero Dirichlet boundary conditions on the sides $u = u_\delta$ and $v = \pi/2$ of R_s, and zero Neumann condition on the side $v = 0$. On the right side of the rectangle its boundary values satisfy

$$k_\delta(s + iv) = \left[1 - e^{4j_{m_1}(u_\delta - s)}\right] \hat{J}_{m_1}\left(2j_{m_1}v/\pi\right) \leq \hat{J}_{m_1}\left(2j_{m_1}v/\pi\right) \leq \hat{J}_{m_1}(0),$$

for all sufficiently large s. Together with (4.36), we see that we can choose a fixed small, but positive, b_1 such that

$$b_1 k_\delta(s+iv) \leq b_1 \hat{J}_{m_1}(0) \leq 1/4 \leq k(s+iv),$$

for all sufficiently large s. Thus, $b_1 k_\delta - k \leq 0$ on the three sides $u = u_\delta$, $u = s$, $v = \pi/2$ of R_s, while $b_1 k_\delta - k$ satisfies a zero Neumann condition on the side $v = 0$. Since $L[b_1 k_\delta - k] = b_1 L[k_\delta] \geq 0$ in R_s, we conclude from the Maximum Principle, Lemma 6, that $b_1 k_\delta - k \leq 0$ in R_s. This leads to a lower bound for $k(u_\delta + 1)$ since

$$\begin{array}{rcl} k(u_\delta+1) & \geq & b_1 k_\delta(u_\delta+1) \\ & = & 2b_1 e^{2j_{m_1}u_\delta/\pi} \sinh(2j_{m_1}/\pi) e^{-2j_{m_1}s/\pi} \hat{J}_{m_1}(0) \\ & \geq & b_2 e^{-2j_{m_1}s/\pi}. \end{array}$$

By the Harnack inequality, $k(0) \geq b_3 k(u_\delta + 1)$, for a constant that does not depend on s. Setting $b_4 = b_3 b_2$,

$$k(0) \geq b_3 k(u_\delta+1) \geq b_4 \exp\left[-\frac{2j_m}{\pi}s - \frac{\epsilon}{2}s\right].$$

As b_4 does not depend on s, we have $b_4 \geq e^{-\epsilon s/2}$ for all sufficiently large s. The proof of Proposition 6 is complete.

□

4.2.6 Lower bound for harmonic measure

The point 0 in the strip S corresponds to the point $(1,0)$ in the region $\mathcal{P}_\alpha^n$ under the transformations in Subsection 4.2.1. Thus,

$$P_{(1,0)}\{|B_{\tau_\alpha}| > t\} = k(0),$$

where the function k satisfies the partial differential equation (4.27) of Proposition 5 and the boundary conditions (4.28), (4.29) and (4.30) with $s = s(t)$ as given by (3.6). Thus Proposition 6 leads directly to the following lower bound for harmonic measure in the parabola-shaped region $\mathcal{P}_\alpha^n$.

Proposition 7. *Suppose that ϵ is positive. There exists a constant C_2 depending on ϵ, n, A and α such that, for $t > C_2$,*

$$P_{(1,0)}\{|B_{\tau_\alpha}| > t\} \geq \exp\left[-\frac{\sqrt{\lambda_1}}{A(1-\alpha)}[1+\epsilon]\, t^{1-\alpha}\right]$$

4.3 Concluding remarks

The distributional inequalities in Propositions 3 and 7 lead immediately to the limit (1.12), and to Theorem 3 by following the steps in the proof of Theorem 2 in Section 3.3.

It is natural to hope that the machinery constructed in Section 4.2 would lead to an upper bound for harmonic measure, and not only to a lower bound, thus rendering the use of the Carleman method and Section 4.1 unnecessary. In fact, there is no mention in Section 4.2 of bounds of any kind until Section 4.2.5. However, we have been unable to prove a counterpart for Proposition 6 involving an upper bound for $k(0)$.

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